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THE $N=2$ SUPER YANG-MILLS LOW-ENERGY EFFECTIVE ACTION AT TWO LOOPS

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Abstract

We have carried out a two loop computation of the low-energy effective action for the four-dimensional $N = 2$ supersymmetric Yang-Mills system coupled to hypermultiplets, with the chiral superfields of the vector multiplet lying in an abelian subalgebra. We have found a complete cancellation at the level of the integrands of Feynman amplitudes, and therefore the two loop contribution to the action, effective or Wilson, is identically zero.

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Following the work by Seiberg and Witten [1] on the construction of the *exact* low-energy effective action for the $N = 2$ supersymmetric Yang-Mills $SU(2)$ theory in its Coulomb phase, considerable attention has been given to the study of effective actions for $N = 2$ theories. This construction starts from a classical action which is invariant under a global $U(1)$ symmetry (R invariance) and a local $SU(2)$ symmetry. By giving a nonvanishing vacuum expectation value to the chiral superfields of the $N = 2$ vector multiplet one both breaks the global invariance and induces a spontaneous symmetry breaking of $SU(2)$ to $U(1)$. The set of inequivalent vacua, parametrized by the vacuum expectation value, determines the "moduli space"; one aims then to obtain an effective lagrangian description of the theory for any value of the modulus. While perturbative calculations are reliable in regions of the moduli space corresponding to weak coupling, in general quantum computations are difficult in the strong coupling regime. The major breakthrough in the work by Seiberg and Witten has been the realization that the complete non-perturbative answer could be actually reconstructed if the perturbative contribution were known exactly.

In fact, $N = 2$ supersymmetry fixes the general form of the low energy effective action (i.e. that part of the effective action which is leading for vanishing momenta): it is given in $N = 2$ superspace by a chiral integral of a holomorphic function $\mathcal{F}(W)$, where W is the $N = 2$ gauge superfield strength of the unbroken $U(1)$. In a $N = 1$ superfield description it has the general form [2]

$$S_{eff} = \frac{1}{16\pi} \mathcal{I}m \left[\int d^4x d^2\theta d^2\bar{\theta} \bar{\phi} \mathcal{F}_\phi(\phi) + \int d^4x d^2\theta \mathcal{F}_{\phi\phi}(\phi) (\frac{1}{2} W^\alpha W_\alpha) \right] . \quad (1)$$

where ϕ denotes the $N = 1$ chiral superfield and W_α the $N = 1$ gauge superfield strength of the $N = 2$, $U(1)$ gauge multiplet. One then has to compute $\mathcal{F}(\phi)$. The work in ref. [1] is based in part on the fact that the perturbative contribution to the low energy effective action, i.e., to $\mathcal{F}(\phi)$, is one-loop exact [3]. This simple result follows essentially from general arguments based on properties of the Wilson action [4], and the fact that the $N = 2$ Yang-Mills theory has an exact one-loop beta function. While this last feature is by now well established, the first point is more delicate and subtle. The Wilson action S_W differs in general from the effective action Γ because in the former one excludes infrared effects by construction, i.e., all loop-momenta are integrated down to an infrared cutoff. In theories with massive particles only, there is no important difference between S_W and Γ ; however, when massless particles are present one has to keep this distinction in mind.

For the low energy effective theory of $N = 2$ Yang Mills one would argue that S_W and Γ are equivalent since the theory does not contain non-abelian gauge interactions and therefore it is well behaved in the infrared. On the other hand it seems natural to ask the question of what happens at the origin of the moduli space, where the $SU(2)$ symmetry is restored and the non-abelian gauge bosons become massless. Since the exact knowledge of the perturbative contribution is crucial for the subsequent

determination of the complete nonperturbative result, and to clarify the issue of effective action versus Wilson action, we have studied the situation at the two loop level.

In this Letter we describe an explicit, two-loop supergraph calculation for the low energy effective action. We have found that *complete* cancellation occurs at the level of the integrands in the Feynman momentum integrals, without the need of any infrared or ultraviolet cutoffs to regularize the theory. This unambiguously checks the complete equivalence of S_W and Γ for the case under consideration. Our calculation follows the method used in ref. [5] where nonholomorphic contributions to the effective action at the one-loop level were computed. In contradistinction to that work, we focus our attention on the low-energy effective action of the massless fields by restricting the (external) chiral superfield component of the $N = 2$ vector multiplet to an abelian subalgebra, as in the Seiberg-Witten analysis.

The classical action for $N = 2$ supersymmetric systems written in $N = 1$ super-space is

$$S_{class} = \frac{1}{4g^2} \left[\int d^4x d^2\theta \operatorname{tr}(\frac{1}{2}W^\alpha W_\alpha) + \int d^4x d^4\theta \operatorname{tr}(\bar{\Phi} e^{-V} \Phi e^V) \right] \\ + \int d^4x d^4\theta (\bar{Q} e^V Q + \tilde{Q} e^{-V} \tilde{\bar{Q}}) + \left(i \int d^4x d^2\theta \tilde{Q} \Phi Q + h.c. \right), \quad (2)$$

The superfields $W_\alpha \equiv i\bar{D}^2(e^{-V}D_\alpha e^V)$ and Φ are $N = 1$ superfield components of the chiral $N = 2$ superfield W that describes the $N = 2$ vector multiplet. V , Φ and $\bar{\Phi}$ are Lie-algebra valued in the adjoint representation, i.e. $V = V^a T_a$, $\Phi = \Phi^a T_a$, etc., with $[T_a, T_b] = if_{ab}^c T_c$ and $\operatorname{tr} T_a T_b = K\delta_{ab}$. The chiral $N = 1$ superfields Q and \tilde{Q} together describe $N = 2$ hypermultiplets. Q and \tilde{Q} are in mutually conjugate representations R and \tilde{R} . We shall explicitly consider the $SU(2)$ case, with $f_{abc} = \epsilon_{abc}$ and $(T_a)_{kl} = i\epsilon_{kal}$ for the adjoint representation and $(T_a)_{kl} = \frac{1}{2}(\sigma_a)_{kl}$ for the fundamental representation.

In addition, after quantizing the vector multiplet we obtain the ghost lagrangian

$$S_{ghost} = \int d^4x d^4\theta \operatorname{tr} \left[\bar{c}'c - c'\bar{c} + \frac{1}{2}(c' + \bar{c}')[V, c + \bar{c}] + \dots \right] \quad (3)$$

with the ghosts in the adjoint representation. We use the notations and conventions of *Superspace* [6].

We perform perturbative calculations by making a quantum-background splitting $\Phi \rightarrow \Phi + \phi$, $\bar{\Phi} \rightarrow \bar{\Phi} + \bar{\phi}$ and, for the $SU(2)$ case considered here, we choose the background fields to lie in the (abelian) “ z ” direction,

$$\phi = (0, 0, \phi) \quad , \quad \bar{\phi} = (0, 0, \bar{\phi}) \quad (4)$$

We consider all diagrams with external ϕ and $\bar{\phi}$. Contributions come from internal lines corresponding to the (Q, \tilde{Q}) hypermultiplets and the $N = 2$ vector multiplet itself. Since we are interested in the low-energy effective action, when doing D -algebra, we drop terms with spinor or space-time derivatives on the external lines. Also, as in ref. [5] we work in Landau gauge for the $N = 1$ vector multiplet. This choice is particularly advantageous: it amounts to dropping all cubic vertices of the form $\text{tr}(\bar{\Phi}[V, \phi])$ and $\text{tr}(\bar{\phi}[V, \Phi])$ since the $D^\alpha \bar{D}^2 D_\alpha$ factor carried by the V -propagator annihilates the D^2 or \bar{D}^2 factors from the quantum $\bar{\Phi}$ or Φ line (and we are not considering terms where the spinor derivatives are integrated by parts onto the background). Thus for the quantum V and Q superfields one can define effective propagators obtained by summing over arbitrary numbers of insertions of the external lines ϕ . We have then in the vector sector

$$\begin{aligned}
\langle V^a V^b \rangle &= -\frac{4g^2}{K} \begin{pmatrix} \frac{1}{p^2 + \phi\bar{\phi}} & 0 & 0 \\ 0 & \frac{1}{p^2 + \phi\bar{\phi}} & 0 \\ 0 & 0 & \frac{1}{p^2} \end{pmatrix}^{ab} \frac{D^\alpha \bar{D}^2 D_\alpha}{p^2} \delta^{(4)}(\theta - \theta') \\
\langle \bar{\phi}^a \phi^b \rangle &= \frac{4g^2}{K} \delta^{ab} \frac{1}{p^2} \delta^{(4)}(\theta - \theta') \\
\langle \bar{c}^a c^b \rangle &= \langle \bar{c}^a c'^b \rangle = \frac{4g^2}{K} \delta^{ab} \frac{1}{p^2} \delta^{(4)}(\theta - \theta')
\end{aligned} \tag{5}$$

For the hypermultiplet in the adjoint representation we find

$$\begin{aligned}
\langle \bar{Q}^a Q^b \rangle &= \frac{1}{K} \begin{pmatrix} \frac{1}{p^2 + \phi\bar{\phi}} & 0 & 0 \\ 0 & \frac{1}{p^2 + \phi\bar{\phi}} & 0 \\ 0 & 0 & \frac{1}{p^2} \end{pmatrix}^{ab} \delta^{(4)}(\theta - \theta') \\
\langle Q^a \tilde{Q}^b \rangle &= -\frac{1}{K} \begin{pmatrix} \frac{1}{p^2 + \phi\bar{\phi}} & 0 & 0 \\ 0 & \frac{1}{p^2 + \phi\bar{\phi}} & 0 \\ 0 & 0 & \frac{1}{p^2} \end{pmatrix}^{ac} \epsilon^{c3b} \bar{\phi} \frac{D^2}{p^2} \delta^{(4)}(\theta - \theta')
\end{aligned} \tag{6}$$

whereas in the fundamental representation

$$\begin{aligned}
\langle \bar{Q}^i Q^j \rangle &= \frac{4\delta^{ij}}{4p^2 + \phi\bar{\phi}} \delta^{(4)}(\theta - \theta') \\
\langle Q^i \tilde{Q}^j \rangle &= -i(\sigma^3)^{ij} \frac{2\bar{\phi}}{p^2(4p^2 + \phi\bar{\phi})} D^2 \delta^{(4)}(\theta - \theta')
\end{aligned} \tag{7}$$

As usual, factors of \bar{D}^2 and D^2 appear at the ends of chiral and antichiral lines.

The one-loop contribution is by now a standard result. One obtains a divergent contribution which simply leads to a renormalization of the coupling constant g in front of the action in (2), as well as a holomorphic contribution to the function \mathcal{F} given by

$$\mathcal{F}(\phi) = \frac{-i}{2\pi} \left(\text{tr}_R(\phi^2 \ln \frac{\phi^2}{\mu^2}) - \text{tr}_{ad}(\phi^2 \ln \frac{\phi^2}{\mu^2}) \right) \quad (8)$$

where μ is the renormalization scale, and the two terms come from the hypermultiplet and vector sectors respectively. At one loop the ghosts do not contribute.

We turn now to the two-loop calculation. The general strategy is to draw all possible supergraphs, evaluate first group theory factors in order to drop immediately the vanishing ones, then perform the D -algebra and assemble the result in the form of momentum integrals. We list below separately the contributions from diagrams which are not zero for group theory or D -algebra reasons.

We consider first the pure $N = 2$ vector multiplet case. The relevant supergraphs are given in Fig.1.

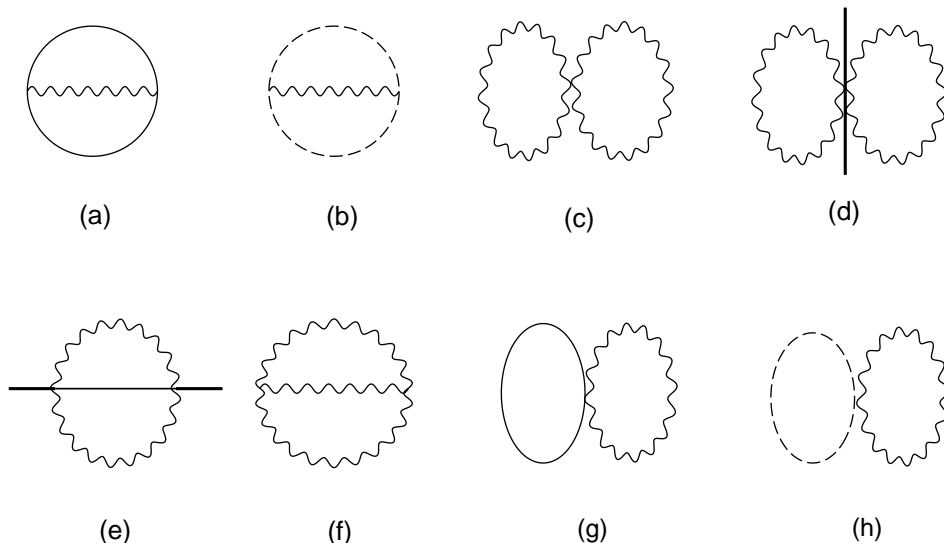


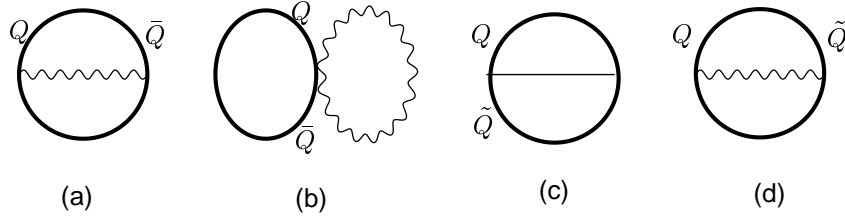
Fig.1. Two-loop diagrams for the $N = 2$ Yang-Mills sector;
solid, wavy, and dashed lines represent Φ , V , and ghost lines, resp.

Note that in addition to the external ϕ lines which give rise to the effective masses and are not indicated, Figs. 1d,e contain some vertices involving explicit external ϕ lines. After D -algebra, we obtain the following contributions, all multiplied by the common factor

$$\frac{4g^2}{K} \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 p}{(2\pi)^4} d^4 \theta$$

$$\begin{aligned}
1a &: -\frac{2p \cdot k}{p^2 k^2 (p+k)^2} \left[\frac{2}{(p+k)^2 + \phi\bar{\phi}} + \frac{1}{(p+k)^2} \right] \\
1b &: \frac{p \cdot k}{p^2 k^2 (p+k)^2} \left[\frac{2}{(p+k)^2 + \phi\bar{\phi}} + \frac{1}{(p+k)^2} \right] \\
1c &: \frac{1}{3k^2} \left[\frac{1}{(p^2 + \phi\bar{\phi})(k^2 + \phi\bar{\phi})} + \frac{1}{p^2(k^2 + \phi\bar{\phi})} + \frac{1}{k^2(p^2 + \phi\bar{\phi})} \right] \\
1d &: \frac{\phi\bar{\phi}}{3k^2 p^2} \left[\frac{4}{(p^2 + \phi\bar{\phi})(k^2 + \phi\bar{\phi})} + \frac{1}{p^2(k^2 + \phi\bar{\phi})} \right] \\
1e &: -\frac{\phi\bar{\phi} p \cdot k}{p^2 k^2 (p+k)^2} \left[\frac{2}{(p^2 + \phi\bar{\phi})(k^2 + \phi\bar{\phi})} + \frac{1}{p^2(k^2 + \phi\bar{\phi})} \right] \\
1f &: \frac{p^2 k^2 - p \cdot k (p+k)^2}{2p^2 k^2} \left[\frac{1}{p^2(k^2 + \phi\bar{\phi})[(p+k)^2 + \phi\bar{\phi}]} \right. \\
&\quad \left. + \frac{1}{k^2(p^2 + \phi\bar{\phi})[(p+k)^2 + \phi\bar{\phi}]} + \frac{1}{(p+k)^2(k^2 + \phi\bar{\phi})(p^2 + \phi\bar{\phi})} \right] \\
1g &: -\frac{2}{k^2 p^2} \left[\frac{2}{k^2 + \phi\bar{\phi}} + \frac{1}{k^2} \right] \\
1h &: \frac{2}{3} \frac{1}{k^2 p^2} \left[\frac{2}{k^2 + \phi\bar{\phi}} + \frac{1}{k^2} \right] \tag{9}
\end{aligned}$$

If hypermultiplets are present, we get additional contributions presented in Fig. 2. In the result from Figs. 2a,b we have included similar contributions from \tilde{Q} .



*Fig.2. Two-loop diagrams from the hypermultiplet sector;
heavy solid lines represent the hypermultiplet.*

If in the adjoint representation, we obtain

$$\begin{aligned}
2a &: -\frac{4p \cdot k}{(p+k)^2} \left[\frac{1}{(p^2 + \phi\bar{\phi})(k^2 + \phi\bar{\phi})(p+k)^2} + \frac{2}{(p^2 + \phi\bar{\phi})k^2[(p+k)^2 + \phi\bar{\phi}]} \right] \\
2b &: -\frac{4}{k^2} \left[\frac{1}{(p^2 + \phi\bar{\phi})(k^2 + \phi\bar{\phi})} + \frac{1}{p^2(k^2 + \phi\bar{\phi})} + \frac{1}{k^2(p^2 + \phi\bar{\phi})} \right] \\
2c &: \frac{2}{(p+k)^2} \left[\frac{1}{(p^2 + \phi\bar{\phi})(k^2 + \phi\bar{\phi})} + \frac{1}{p^2(k^2 + \phi\bar{\phi})} + \frac{1}{k^2(p^2 + \phi\bar{\phi})} \right] \\
2d &: \frac{4\phi\bar{\phi}}{(p^2 + \phi\bar{\phi})(k^2 + \phi\bar{\phi})(p+k)^4}
\end{aligned} \tag{10}$$

If in the fundamental representation,

$$\begin{aligned}
2a &: -\frac{16p \cdot k}{(4p^2 + \phi\bar{\phi})(4k^2 + \phi\bar{\phi})(p+k)^2} \left[\frac{2}{(p+k)^2 + \phi\bar{\phi}} + \frac{1}{(p+k)^2} \right] \\
2b &: -\frac{4}{(4p^2 + \phi\bar{\phi})k^2} \left[\frac{2}{k^2 + \phi\bar{\phi}} + \frac{1}{k^2} \right] \\
2c &: \frac{24}{(4p^2 + \phi\bar{\phi})(4k^2 + \phi\bar{\phi})(p+k)^2} \\
2d &: \frac{4\phi\bar{\phi}}{(4p^2 + \phi\bar{\phi})(4k^2 + \phi\bar{\phi})(p+k)^2} \left[\frac{1}{(p+k)^2} - \frac{2}{(p+k)^2 + \phi\bar{\phi}} \right]
\end{aligned} \tag{11}$$

It is then a matter of straightforward algebra to check that separately, the contributions in (9), (10) and (11) add up to zero. We have thus verified by an explicit supergraph calculation that the two-loop contribution to the low-energy $N = 2$ effective action for the pure Yang-Mills theory, or for the Yang-Mills theory coupled to hypermultiplets, vanishes identically.

Some comments are in order:

a. The cancellation of the contributions to the low energy effective action is complete, including tadpole-type $1/p^2$ integrals that we would normally discard in dimensional regularization; there is no need for some low-energy cutoff. In principle, we should imagine some ultraviolet regularization procedure (such as dimensional regularization) which permitted us to shift loop momenta as needed, in order to obtain the cancellation. Presumably, even this is not necessary.

b. The removal of divergences of the $N = 2$ effective action amounts to a one-loop renormalization (in dimensional regularization, say) $1/g^2 \rightarrow 1/g^2 + c/\epsilon$. Consequently, the coupling constant g itself gets renormalized at all orders, $g^2 \rightarrow g^2 - cg^4/\epsilon + c^2g^6/\epsilon^2 + \dots$, as expected from the 't Hooft pole equations.

c. At the one-loop level, in ref. [5] we have found some non-holomorphic corrections to the effective action when the external chiral superfields ϕ do not lie in an abelian subalgebra. It is of interest to examine the corresponding situation at the two-loop level.

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